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Classical q-deformed dynamics

A. Lavagno[1,2], A.M. Scarfone[1,3], and P. Narayana Swamy[4,a]

[1] Dipartimento di Fisica, Politecnico di Torino, Italy
[2] Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Torino, Italy
[3] Istituto Nazionale di Fisica della Materia (CNR-INFM), Sezione di Torino, Italy
[4] Southern Illinois University, Edwardsville, IL 62026, USA

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Abstract. On the basis of the quantum q-oscillator algebra in the framework of quantum groups and non-commutative q-differential calculus, we investigate a possible q-deformation of the classical Poisson bracket in order to extend a generalized q-deformed dynamics in the classical regime. In this framework, classical q-deformed kinetic equations, Kramers and Fokker-Planck equations, are also studied.

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1 Introduction

The study of quantum algebras and quantum groups has attracted a lot of interest in the last few years, and stimulated intensive research in several physical fields in view of a broad spectrum of applications, ranging from cosmic strings and black holes to the fractional quantum Hall effect and high-T_c superconductors [1].

From the seminal work of Biedenharn [2] and Macfarlane [3] it is clear that the q-calculus, originally introduced at the beginning of last century by Jackson [4] in the study of the basic hypergeometric function, plays a central role in the representation of the quantum groups [5]. In fact it has been shown that it is possible to obtain a “coordinate” realization of the Fock space of the q-oscillators by using the deformed Jackson derivative (JD) or the so-called q-derivative operator [6–8].

In this paper we want to introduce a q-deformation of the PB (q-PB) in order to define a generalized q-deformed dynamics in a q-commutative phase-space. For this purpose we begin with the observation that the creation and annihilation operators in the quantum q-deformed $\mathrm{SU}_q(2)$ algebra corresponds classically to q-commuting coordinates in a q-phase space and that the commutation relation between the standard quantum operators corresponds classically to the Poisson bracket (PB).

The motivation for our goal lies in the fact that a full understanding of the physical origin of q-deformation in classical physics is still lacking because it is not clear if there exists a classical counterpart to the q-deformed quantum mechanics inspired by the study of quantum groups. The problem of a possible q-deformation of classical mechanics was dealt with in reference [9] where a q-PB was obtained starting from a point of view different from the one adopted in this paper. In order to introduce the classical correspondence of the quantum q-oscillator, we shall follow the main approach based on the following idea. The (undeformed) quantum commutation relations are invariant under the action of SU(2) and, as a consequence, the q-deformed commutation relations are invariant under the action of $\mathrm{SU}_q(2)$. Analogously, since the (undeformed) PB is invariant under the action of the symplectic group Sp(1), we have to require that q-PB must satisfy invariance under the action of the q-deformed symplectic group $\mathrm{Sp}_q(1)$.

2 Non-commutative differential calculus

Since the creation and annihilation operators in the quantum q-deformed $\mathrm{SU}_q(2)$ algebra correspond classically to non-commuting coordinates in a q-phase-space, in this section we introduce the q-deformed plane which is generated by the non-commutative elements $\hat{x}$ and $\hat{p}$ fulfilling the relation [10]

$$\hat{p}\,\hat{x} = q\,\hat{x}\,\hat{p}, \tag{1}$$

which is invariant under $\mathrm{GL}_q(2)$ transformations. Henceforward, for simplicity, we shall limit ourselves to consider the two-dimensional case.

From equation (1) the q-calculus on the q-plane can be obtained formally through the introduction of the q-derivatives $\hat{\partial}_x$ and $\hat{\partial}_p$ [11]

$$\hat{\partial}_p\,\hat{p} = \hat{\partial}_x\,\hat{x} = 1, \tag{2}$$

$$\hat{\partial}_p\,\hat{x} = \hat{\partial}_x\,\hat{p} = 0. \tag{3}$$

[a] e-mail: pswamy@siue.edu

They fulfill the q-Leibniz rule

$$\hat{\partial}_p\,\hat{p} = 1 + q^2\,\hat{p}\,\hat{\partial}_p + (q^2-1)\,\hat{x}\,\hat{\partial}_x, \tag{4}$$
$$\hat{\partial}_p\,\hat{x} = q\,\hat{x}\,\hat{\partial}_p, \tag{5}$$
$$\hat{\partial}_x\,\hat{p} = q\,\hat{p}\,\hat{\partial}_x, \tag{6}$$
$$\hat{\partial}_x\,\hat{x} = 1 + q^2\,\hat{x}\,\hat{\partial}_x, \tag{7}$$

together with the q-commutative derivative

$$\hat{\partial}_p\,\hat{\partial}_x = q^{-1}\,\hat{\partial}_x\,\hat{\partial}_p. \tag{8}$$

It is easy to see that in the $q \to 1$ limit one recovers the ordinary commutative calculus. Let us outline the asymmetric mixed derivative relations, equations (4) and (7), in $\hat{x}$ and in $\hat{p}$. These properties arise directly from the non-commutative structure of the q-plane defined in equation (1).

We recall now that the most general function on the q-plane can be expressed as a polynomial in the q-variables $\hat{x}$ and $\hat{p}$

$$f(\hat{x},\,\hat{p}) = \sum_{i,j} c_{ij}\,\hat{x}^i\,\hat{p}^j, \tag{9}$$

where we have assumed the $\hat{x}$-$\hat{p}$ ordering prescription (it can always be accomplished by means of Eq. (1)). Thus, taking into account equations (4)–(7), we obtain the action of the q-derivatives on the monomials

$$\hat{\partial}_x(\hat{x}^n\,\hat{p}^m) = [n]_q\,\hat{x}^{n-1}\,\hat{p}^m, \tag{10}$$
$$\hat{\partial}_p(\hat{x}^n\,\hat{p}^m) = [m]_q\,q^n\,\hat{x}^n\,\hat{p}^{m-1}, \tag{11}$$

where we have introduced the q-basic number

$$[n]_q = \frac{q^{2n}-1}{q^2-1}. \tag{12}$$

A realization of the above q-algebra and its q-calculus can be accomplished by the replacements [12]

$$\hat{x} \to x, \tag{13}$$
$$\hat{p} \to p\,D_x, \tag{14}$$
$$\hat{\partial}_x \to \mathcal{D}_x, \tag{15}$$
$$\hat{\partial}_p \to \mathcal{D}_p\,D_x, \tag{16}$$

where

$$D_x = q^{x\,\partial_x}, \tag{17}$$
$$D_x f(x,\,p) = f(q\,x,\,p), \tag{18}$$

is the dilatation operator along the x direction (reducing to the identity operator for $q \to 1$), whereas

$$\mathcal{D}_x = \frac{q^{2\,x\,\partial_x}-1}{(q^2-1)\,x}, \tag{19}$$
$$\mathcal{D}_p = \frac{q^{2\,p\,\partial_p}-1}{(q^2-1)\,p}, \tag{20}$$

are the JD with respect to x and p. Their action on an arbitrary function $f(x,\,p)$ is

$$\mathcal{D}_x\,f(x,\,p) = \frac{f(q^2\,x,\,p) - f(x,\,p)}{(q^2-1)\,x}, \tag{21}$$
$$\mathcal{D}_p\,f(x,\,p) = \frac{f(x,\,q^2\,p) - f(x,\,p)}{(q^2-1)\,p}. \tag{22}$$

Therefore, as a consequence of the non-commutative structure of the q-plane, in this realization the $\hat{x}$ coordinate becomes a c-number and its derivative is the JD whereas the $\hat{p}$ coordinate and its derivative are realized in terms of the dilatation operator and JD.

3 q-Poisson Bracket and q-symplectic group

With the formulation of the q-differential calculus, we are now able to introduce a q-PB. Since the undeformed PB is invariant under the action of the undeformed symplectic group Sp(1), we will assume as previously stated, as a fundamental point, that the q-PB must satisfy the invariance property under the action of the q-deformed symplectic group $\mathrm{Sp}_q(1)$ with the same value of the deformed parameter q used in the construction of the quantum plane.

Let us start by recalling the classical definition of a 2-Poisson manifold, which is a two dimensional Euclidean space $\mathbb{R}^2$ generated by the position and momentum variables $x \equiv x^1$ and $p \equiv x^2$ and equipped with a PB. By introducing $f(x,\,p)$ and $g(x,\,p)$, two arbitrary smooth functions, the PB is defined as [13]

$$\Big\{f,\,g\Big\} = \partial_x f\,\partial_p g - \partial_p f\,\partial_x g. \tag{23}$$

Equation (23) can be expressed in a compact form

$$\Big\{f,\,g\Big\} = \partial_i f\,J^{ij}\,\partial_j g, \tag{24}$$

where J^{ij} are the entries of the unitary symplectic matrix J given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

Remarkably, equation (24) does not change under the action of a symplectic transformation Sp(1) on the phase-space. As is well known equation (24) can also be expressed as

$$\Big\{f,\,g\Big\} = \{x^i,\,x^j\}\,\partial_i f\,\partial_j g, \tag{26}$$

so that, if we know the PB between the generators x^i we can compute the PB between any pair of functions f and g.

By requiring that the q-PB must be invariant under the action of the q-symplectic group $\mathrm{Sp}_q(1)$, we are lead to introduce the following q-deformed PB between the q-generators $\hat{x}^i$ [14]

$$\Big\{\hat{x}^i,\,\hat{x}^j\Big\}_q = \hat{\partial}_x\,\hat{x}^i\,\hat{\partial}_p\,\hat{x}^j - q^2\,\hat{\partial}_p\,\hat{x}^i\,\hat{\partial}_x\,\hat{x}^j. \tag{27}$$

It is easy to verify the following fundamental relations

$$\Big\{\hat{x},\,\hat{x}\Big\}_q=\Big\{\hat{p},\,\hat{p}\Big\}_q=0, \tag{28}$$

$$\Big\{\hat{x},\,\hat{p}\Big\}_q=1, \tag{29}$$

$$\Big\{\hat{p},\,\hat{x}\Big\}_q=-q^2, \tag{30}$$

which coincide with the one obtained in reference [9]. In particular, from equations (29) and (30) it follows that the q-PB is not antisymmetric. A similar behavior appears also in the quantum q-oscillator theory [2,3].

By means of equations (13)−(16), a realization of our generalized q-PB can be written as

$$\Big\{f,\,g\Big\}_q=\mathcal{D}_x\,f(x,\,p\,D_x)\,\mathcal{D}_p\,g(q\,x,\,p\,D_x) - q^2\,\mathcal{D}_p\,f(q\,x,\,p\,D_x)\,\mathcal{D}_x\,g(x,\,p\,D_x) \tag{31}$$

where f and g are identified with x or p, respectively.

4 q-Deformed kinetic equations

On the basis of the above classical q-deformed theory, we shall now derive the corresponding classical kinetic equations based on the q-calculus. Starting from the realization of the q-algebra, defined in equations (13)−(16), we are able to write the Kramers equation corresponding to the equation of motion for the distribution function $f(x,\,p;\,t)$, in position and momentum space, describing the motion of particles of mass m in an external field $F(x)$ [15]. In the one-dimensional case it can be generalized as follows

$$\frac{\partial f(x,\,p;\,t)}{\partial t}=\Big\{-\frac{p}{m}\mathcal{D}_x D_x-\mathcal{D}_p D_x\,\left[J_1^q(p\,D_x)+F(x)\right] + J_2^q\,(\mathcal{D}_p\,D_x)\,(\mathcal{D}_p\,D_x)\Big\}\,f(x,\,p;\,t), \tag{32}$$

where $J_1^q(p\,D_x)$ and J_2^q are the drift and diffusion coefficients, respectively. Specifying the action of the dilatation operator D_x along the x direction, the above Kramers equation can be written as

$$\frac{\partial f(x,\,p;\,t)}{\partial t}=-\frac{p}{m}\,\mathcal{D}_x\,f(q\,x,\,p;\,t)-\mathcal{D}_p\,[J_1^q(p\,D_x) + F(q\,x)]f(q\,x,\,p;\,t)]+J_2^q\,\mathcal{D}_p^2\,f(q^2\,x,\,p;\,t), \tag{33}$$

where $\mathcal{D}_p^2$ means the double application of the JD in the momentum space. Without any external force, for a homogeneous system undergoing a constant diffusion, the above generalized Kramers equation reduces to the following q-deformed Fokker-Planck equation

$$\frac{\partial f(p;\,t)}{\partial t}=\mathcal{D}_p\Big[-J_1^q(p)+J_2^q\,\mathcal{D}_p\Big]\,f(p;\,t). \tag{34}$$

If we postulate a generalized Brownian motion in a q-deformed classical dynamics by the following definition of the drift and diffusion coefficients

$$J_1^q(p)=-\gamma\,p\,\left(\frac{q^2\,D_p+1}{2}\right), \qquad J_2^q=\gamma\,m\,kT\,, \tag{35}$$

where γ is the friction constant, T is the temperature of the system and D_p is the dilatation operator in the momentum space, the stationary solution $f_{\rm st}(p)$ of the above Fokker-Planck equation can be obtained as a solution of the following stationary q-differential equation

$$\mathcal{D}_p f(p)=-\frac{p}{2\,m\,kT}\left[q^2\,f(qp)+f(p)\right]. \tag{36}$$

It is easy to show that the solution of the above equation can be written as

$$f_{\rm st}(p)=E_q\left[-\frac{p^2}{2m\,kT}\right], \tag{37}$$

where $E_q[x]$ is the q-deformed exponential function, well-known in q-calculus [5], defined in terms of the series

$$E_q[x]=\sum_{k=0}^{\infty}\frac{x^k}{[k]_q!} \tag{38}$$

where $[k]_q!$ is the q-basic factorial [5] defined as $[k]_q!=[k]_q\,[k-1]_q\cdots[1]_q$.

5 Conclusions

We have shown that q-calculus can play a crucial role in the formulation of a generalized q-classical theory, defined by means of the introduction of a q-PB. In analogy with quantum group invariance properties of the quantum q-oscillator theory, the q-PB has been defined by assuming the invariance under the action of $\mathrm{Sp}_q(1)$ group with its derivatives acting on the q-deformed non-commutative plane invariant under $\mathrm{Gl}_q(2)$ transformations. Therefore such a classical q-deformation theory can be seen as the analogue of q-oscillator deformation in the quantum theory. In this framework, we have studied the classical q-deformed kinetic equations, Kramers and Fokker-Planck equations and we have found, as a stationary solution, the well-known q-deformed exponential function defined in terms of a series. This opens the possibility of introducing a classical counterpart of the quantum q-deformations. We expect that such a classical q-deformed dynamics can be very relevant in several physical applications such as, in the resolution of integrable systems. Further important applications would be in the formulation of effective theory of complex many-body systems and in the framework of a generalized thermostatistics [16] in a manner similar to what classical Tsallis [17] or Kaniadakis [18] thermostatistics does with respect to the Boltzmann-Gibbs theory.

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